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ANCIENT INDIAN DISCOVERIES : NUMBERS AND ZERO

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NEW DELHI - 110 002

DR. N. GOPALAKRISHNAN
Scientist

INDIAN INSTITUTE OF SCIENTIFIC HERITAGE
THIRUVANANTHAPURAM - 695 018

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INTRODUCTION

Ŗavindranath Tagore on the glory of Bharath (J.C. Bose)

Young image of what old Rishi of India
Art thou, O Arya, savant Jagadish?
What unseen hermitage hast though raised up
From 'neath the dry dust of this city of stone?
Amidst the crowd's mad turmoil, whence hast though
That peace in which thou in an instant stoodst
Alone at the deep centre of all things-
Where dwells the One along in Sun, Moon, flolwers,
In leaves, and beasts and birds and dust and stones,
-where still one sleepless Life on its own lap
Rocks all things with a wordless melody,
All things that move or that seem motionless!
While we were drunk with the remote and vain
Dead glories of our past, - in alien dress
Walking and talking in an alien tongue,
in the caricature of other men-
Their style, their bearing- While we should, yelled.
Frog-like with swollen throat in our dark well,
O' in what vast remoteness were though then?
Where didst thou spread thy hush's and lonely mat-
Thy mat of mediation? Thou, thy mind
curdling into calm gravity, didst plunge

In thy great quest after the viewless ray,
beyond the utmost borders of this world
Of visible form, there where the Rishies old
Oped, and passed in beyond the lion-gates
Of the mainfold and stood before the One,
silent in awe and wonder, with joined hands!
O Hermit, call thou in the authentic words
of that old hymn called Sama: "Rise! Awake!"
Call to man who boasts his Sastric lore
From vain pedantic wringlings profitless,
Call to that foolish braggart to come forth
Out one the face of nature, this broad earth
Send forth this call unto thy scholar band;
Together round thy sacrifice of fire
Let them all gather,. So may our India,
Our ancient Land, unto herself return
O once again return to steadfast work.
To duty and devotion, to her trance
Of earnest meditation; let her sit
Once more unruffled greedless, strifeless, pure
O once again upon her lofty seat
And platform, teacher of all other lands

(Jagathguru status)

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In ancient India mathematics had an esteemed status among all branches of knowledge. Aptly, ganita was compared to the jewel on the hood of serpents and with eyes among the sense organs. Jyothisha is one among the six Vedangas of Vedic and related literature. Jyothisha include astronomy and mathematics. In modern times also the students of mathematics are expected to learn astronomy as a part of their curriculum. In Vedanga Jyothisha, it is thus said (Yajusha Jyothisham 4):

यथा शिखा मयूराणां नागानां मणयो यथा।
तद्वद् वेदाङ्गशास्त्राणां गणितं मूर्धनि स्थितम्।।

Yathaa sikhaa mayooraanaam naagaanaam manayo yathaa
thaduat vedaangasaasthranaam ganitham moordhani sthitham

Similar to sikha on the head of a peacock and jewel on the hood of a serpent, the status of mathematics is on the forehead of Vedanga sutras.

In modern science, every branch of knowledge studied with the support of mathematics. It is so for all scientific, cultural and social subjects. The ancient Indians had the same opinion on the application of mathematics for various human activities. Thus says Mahaveeracharya (Ganita Sarasangtaha 1-9-ii):

लौकीके वैदिके सामायिकेऽपि यः व्यापारस्तत्र सर्वत्र संख्यानुपयुज्यते!
कामतन्त्रेऽर्थशास्त्रे च गान्धर्वे नाटकेऽपि वा सूपशास्त्रे तथा वैद्ये
वास्तु विद्यायादि वस्तुषु बहुभिर्विप्रलाभैः किं त्रैलोक्यसचराचरैः
यत्किञ्चिद्वस्तु तत्सर्वं गणितेन विना न हि ।।

Lowkeeke vaidike saamayikefpi ya: vyaapaarasthathra sarvathra sankhyaanupayujyate kaamathanthrefrthasaasthre cha gaandharve naatakefpi vaa soopasaastre thatha vaidye vaasthu vidyadi vastushu bhabubhirvipralaabhai: kim thrailokya sacharaacharai: yatkinchivasthu thatsarvam ganithena vinaa na hi

Mathematics is used in all calculations. It has a role to play in all common social activities, Vedic rituals, commercial and transactional activities, in sex, pure and applied sciences, music, dance, drama, cooking, medical sciences, architecture, etc. Other than the above, for better understanding and application of various branches of knowledge, mathematics is used by learned people. It is said that without the application of mathematics, nothing can exist in three worlds.

As mentioned above, modern science has accepted the principle that for every branch of social and scientific studies, statistical evaluations and accounting are inevitable. This supports the ancient Indian view on the application of mathematics.

Mathematics is said to have a systematic beginning in India, which was connected with the development of Vedic rituals. Mathematics is an important part in Kalpasutras, which is known as Sulbasutras. In the Sulbasutra too, a continuous development and refinement on many geometrical calculations had been taking place from the books of earlier origin to those of later origin. The mathematical knowledge has been developing for achieving better accuracy in results and methods for applications. It is a well known scientific temper attribute that every knowledge has to be continuously refined. Thus say scholars on achieving prefection in mathematics.

कालान्तरे तु संस्कारश्चिन्त्यतां गणकोत्तमैः

Kaalanthare thu samskaaraschinthyathaam ganakotthamai:

During the course of time, great mathematicians should refine the subject (rules and equations on the subject).

This principle obviously declare the rational and scientific approach Which is the same as that of modern scientists. It is not correct to believe that in ancient India all that was taught by Gurus and teachers were transferred to generations 'blindly'

without adequate thought input. Both, in theoretical and applied aspects, this approach of refinement of mathematical principles can be seen in concurrence with a steady progress in the depth of knowledge in subject matter. The books on mathematics written from 1000 BC (Sulbasutras) to the later books written in the 18th and 19th centuries AD, this progress in content is clearly visible. The search of modern concepts in mathematics has to commence from the development of number systems.

Numbers :

Rig Veda is the oldest literature, hitherto available, of the human race. It contains fundamentals of the number systems. But a very systematic presentation of numbers can be seen in different texts of Yajurveda (xvii. 24, 25)

एका चमे तिस्रश्चमे पंच चमे सप्त चमे नवचमे एकादश च मे त्रयोदश च मे पञ्चदश चमे सप्तदश चमे नवदश चमे एकविंशतिश्च मे त्रयोविंशतिश्च मे पञ्चविंशतिश्चमे सप्तविंशतिश्चमे नवविंशतिश्चमे एकत्रिंशच्चमे त्रयस्त्रिंशच्चमे......

Eka chame thisraschame pancha chame saptha chame navachame ekaadasa cha me thrayodasa chame panchadasa chame sapthadasachame, navadasa chame ekavimsathischame, thrayovimsathischame panchavimasthi schame sapthavimsathischame navavimsatischame eka thrimsaschame thryasthrimsachha me ....

This is an arithmetic progression of odd numbers starting from 1 and ending in 33, proportionately (with common difference of 2).

Even number progressions are also given in the Yajurveda book (xviii - 23). Presentation of numbers, as a series of multiples of ten is yet another observation noted in Yajurvedic Vajasaneyee samhita (xvii. 2), showing an experienced handling of numbers in mathematics. It gives the proof on the first use of decimal

places in writing numbers as early as 2500 BC (a few historians fix this period for Yajurveda and a few others go further back to 4000 BC).

एकं च दशं च दशंच शतंच शतंच सहस्रंच
सहस्रंचायुतं च अयुतं च नियुतं च नियुतं च प्रयुतं च
प्रयूतंचार्बुदंच समुद्रश्च, मध्यम्चान्तश्च परार्धश्चैता ......

Ekam cha dasam cha dasam cha satamchasatam cha sahasram cha sahasram chaayutham cha ayutam cha niyutham cha niyutham cha prayutham cha prayutham chaarbudam cha samudrascha madhyam chaanthascha paraardhaschaithaa....

One, ten, hundred, thousand, one lac, one crore upto ten thousand crore is given here. Similar presentations are available in Taitireeya samhita (1.5.11.1), which is another recension of Yajurveda itself.

Use of numbers for presenting data on length, breadth and area of the Yajna saala (sacrificial/ritual hall), can be seen in all the four important Sulbasutras namely, Boudhayana, Apastamba, Katyayana and Manava sulbasutras. Some of these books are chronologically belong to the same period as those of Yajurvedas.

Opinion put forth by professor Neugebauer on the mathematical content in these texts attracts attention of the subject experts in this area "From the time of Samhitas, the Vedic Indians used the decimal scale without the use of symbols. The expressions of numbers of the scale, eka, dasa, ... upto 18th power of 10 were given in Sulba sutras....... The successive placing of dasa, sata, sahasra, etc. is obvious proof for the decimal places..." Says Neugebauer, in his celebrated book, 'The Exact Science in Antiquity' (pp 10, 13-14). In the comment 'without the use of symbols' Neugebauer might have meant only that in the text they have not given the symbols. It cannot be taken as an

observation on the non existence of symbols for writing numbers. Because in Rig Veda, it is mentioned 'give thousand cows whose ears are marked with the number eight' (*ashtakarnya*). Marking a number eight shows knowledge in number system.

It is important to note here that when Greeks were using only upto a maximum value Myriad (1000) and Romans, Millie (1000), Indians could go upto 18th power of 10 level during the Vedic period. Dr. Hopkins gives a better picture of the Greek mathematicians of the 1st millennia BC. He says "Before the 6th century BC, all these religious and philosophical ideas of Pythagorus were current in India." The well known philosopher of ancient Greek, Appolonius has mentioned that Pythagorus went to India and was taught by Brahmins, on the geometrical rules. Not much is known about the European knowledge of mathematics after Pythagorus and Euclid, for nearly a thousand years. It has been told that Leonardo Fibonancii of Pisa spread Hindu numerals in Europe. By 1228 AD, he wrote a book focussing on Indian mathematics namely Liber abaci. Historians say that liber abaci is the stepping stone for the west to the modern mathematics. Evidences are many for this inference. The numerical words penta (pancha), hexa (shasta), septa (sapta), octa (ashta), nona (nava), deca (desa).... penta deca (pancha dasa).... octa deca (ashta dasa)..... etc. are still used in the number systems by the Europeans and the English. All these terms are in Vedas.

Albiruni's book written in 1030 AD, namely Tarik al Hind (Chronicles of India) in which he says that "The numeral signs which we (the people of the west) use are derived from the finest forms of the Hindu signs". A glimpse on the development of mathematics in the second half of the first millennia AD, will definitely give an insight on the real Indian contributions, prior to Leonardo Fibonancii.

In Aryabhatteeya 1 to 9th power of 10, places have been mentioned, specifically for the purpose of understanding the rule of writing numbers. (Aryabhateeya 2-2)

एकं दशं च शतं च सहस्रं त्वयुतनियुते तथा प्रयुतम् ।
कोट्यर्बुदं च वृन्दं स्थानात् स्थानं दशगुणं स्यात् ।।

Ekam dasam cha satham cha sahasram thwayuthaniyuthe thathaa prayutham kotyarbudam cha vrundam sthaanath sthaanam dasagunam syaath

One, ten, hundred, thousand, ...to thousand million numbers. From place to place (towards left) each number is ten times to the preceding ones.

Aryabhateeya was famous not only in India but also in Arabia and other countries. Merits of his astronomical and mathematical works were well appreciated the world over. In Arabia the book Aryabhateeya was known as Arjbar - Sindhind which means Aryabhatta's siddhanta. His contributions in mathematics have also been transferred to Arabia and from there to the west. Aryabhatta has also developed another number system which is explained elsewhere in the text.

A systematic presentation of higher/larger numbers can be seen in the texts written after Aryabhateeya. Sreedharacharya in Patiganita (I. 7-8), gives the number from one to ten thousand crore crore i.e 10^{18}, in the order of multiples of ten.

एकं दश शतमस्मात्सहस्रमयुतं ततः परं लक्षम् प्रयुतं कोटिमथार्बुदमब्ज
खर्वं निखर्वं च तस्मान् महासरोजं शङ्कु सरितां पतिं ततस्त्वान्यं
मध्यं परार्द्धमाहुर्यथोत्तरं दशगुणं तज्ज्ञाः ।।

Ekam dasa sathamasmaathsahasramayutham thatha: param laksham. prayutham kotimathaarbudamabja krarvam nikharvam cha thasmaan mahaasarojam sankhu sarithaam pathim

thathasthvaanyam madhyam paraardhamaahur yathottharam dasagunam thajnyaa:

The order of starting the number is eka, dasa..... and ending in madhya, parardha. Each number is stated, ten times the preceeding, by those who have knowledge of mathematics.

Further, higher numbers are given seen in Mahaviracharya's (850 AD) book. It gives the number upto 10^{24} (mahakshobha) in Ganita tilaka (55. 2-3) as follows:

एकं दशं शतं सहस्रं दशसहस्रं लक्षं दशलक्षं कोटि दशकोटि शतकोटि अर्बुद न्यर्बुद खर्व महाखर्व पद्म महापद्म क्षोणि महाक्षोणि शङ्कु महाशङ्कु क्षिति महाक्षिति क्षोभ महाक्षोभ.....

Ekam dasam satham sahasram dasasahasram laksham dasalaksham koti dasakoti sathakoti arbuda nyarbuda kharva mahakharva padma mahapadma kshoni mahaakshoni sanku mahaasanku kshithi mahakshithi kshobha mahakshobha.........

Yallayya (1480 AD) has given in his Aryabhateeya bhashya the last number upto 10^{29} which is known as bhuri

एक दश शत सहस्र अयुत लक्ष प्रयुत कोटि दशकोटि शतकोटि अर्बुद न्यर्बुद खर्व महाखर्व पद्म महापद्म शङ्कु महाशङ्कु क्षोणि महाक्षोणि क्षिति महाक्षिति क्षोभ महाक्षोभ परार्ध सागर अनन्त चिन्त्य भूरि ..

Eka dasa sahasra ayutha laksha prayutha koti dasakoti satakoti arbuda kharva mahakharva padma mahapadma sanku mahasanku kshoni mahakshoni kshithi mahakshithi kshobha mahakshobha paraardha saagara anantha chinthya bhoori ....

Pavalluri Mallikarjuna starts the number series from one and ends in 10^{36}. This number was called the mahabhuri by him in his book. (Ganita sastra)

A point to be noted here is that rarely a few of the same number terms denote different numbers when used by different mathematicians. Yallayya used bhuri for 10^{29} and Mallikarjuna used bhuri for 10^{35}. (In modern mathematics also billion has two values i.e thousand million (USA) and a million million (UK).

It is clear that writing numbers with many place values has been very common in ancient India even during Vedic period and in the first millennia AD. During this period the Europeans and other country men were only entering the doorsteps of the world of mathematics.

Method of Presenting mathematical data

In ancient India, almost all mathematical data like numerical values, theorems, equations etc., have been presented as slokas (poetical stanzas) which is entirely different from the present style followed in modern mathematics. For easy presentation and keeping the chandas (metres) numbers, three number systems (other than the common Sanskrit system) were adopted by astronomers and mathematicians. This appears to be unique to ancient India. Available literature do not give anything similar to this in other civilisations. However for the understanding of ancient Indian mathematical and astronomical contributions knowledge on all the three number systems viz. Aryabhateeya system, Bhootha sankhya system and Katapayadi system. A glance on all the three systems are given below.

Aryabhateeya number system: This number system was used for the first time by Aryabhatta I in his book Aryabhateeya. In the first chapter of the book known as Geetika Padam, this system is used. The basis of this number system is mentioned by him in the second stanza of the first chapter (Aryabhateeya - 1.2)

वर्गाक्षराणि वर्गे/वर्गे/वर्गाक्षराणि कात् ङ्मौ यः

खद्विनवके स्वरा नव वर्गे/वर्गे नवान्त्यवर्गे

Vargaaksharaani vargefvarge fvargaaksharaani kaath ngmow ya:
khadvi navake swaraa nava varge avarge navaanthya varge ya:

The varga (group/class) letters Ka to Ma are to be placed in the varga (square) places (1st, 100th, 10000th.... etc. places) and avarga letters like ya, ra, la, ... have to be placed in avarga places (10th, 1000th,.....etc. places). (varga letters - ka to ma - have value 1,2,3..... upto 25 and avarga letters - ya to ha - value 30, 40, 50.... upto 100). The 'value' in using this number system is like getting the sum of nga and ma (i.e. 5 + 25) = ya = 30. Nine vowels should be used upto nine place values in the varga (and avarga) places. In the varga and avarga letters, beyond the ninth vowel (place), new symbols can be used.

Sanskrit vowels are as follows: i = 100; u = 10000, ru = 1000000 and so on. Example, cha = 6; chi = 600; chu = 60000. Sum of the values of the letters, with or without vowels give the number. As for gi yi nga sa = gi + yi + nga + sa= 300 + 3000 + 5 + 70 = 3375; Similarly njila = nji + la = 1000 + 50 = 1050.

Using this number system small and large numbers can easily be written. Aryabhatta has used this system for presenting the astronomical and mathematical data. For presenting fractions also this number system can conveniently be used. Thus: Nga, nja, n'a, na mamsaka is 1/5, 1/10, 1/15, 1/20 and 1/25. Mixed fractions can be written as; Jhardham (jha is nine; it's half) = 4½. It appears that only Aryabhatta used this number system and no other mathematician/astronomer has used it for their original contributions, but used in their commentaries on Aryabhateeya.

Bhootha Sankhya System : Bhootha sankhya is one of the most commonly used number systems for presenting data on astronomy and mathematics. It can well combine with the Sanskrit number system. It is said that the earliest work in which Bhootha sankhya is found is Pingalacharya's Chanda sastra (200

BC). Bhootha sankhya can be understood from the following explanations, which are quoted from Sankaranarayana's commentary for Laghubhaskareeyam (1-15,16)

चन्द्रश्शीतांशुरिन्दुश्च चन्द्रमा हिमगुः शशी
एवमादीनि नामानि चन्द्रस्य कथितानि च ।।
रूपमित्येतदेकस्य द्वयोरपि च कीर्त्यते
नयनस्य तु नामानि युग्मं युगलमेव च ।।
यमं च यमलं चैव दस्रौ नासत्य एव च
अश्विनोर्नामधेयत्वात् द्विसंख्येति प्रकीर्तिते ।।
अग्निनामानि यान्यत्र गुणो लोकाश्च पुष्कराः
रामो व्रतं त्रयाणांतु कीर्तितानि बुधैस्सदा ।।
वेदपर्यायशब्दाश्च समुद्रस्य तथैव च
कृतश्चेति चतुर्णांच संख्या सद्भिरुदाहृता ।।
इन्द्रियाणि च भूतानि कामदेवेषवस्तथा
वायुपर्याय शब्दाश्च पञ्चानां तु प्रकीर्तिताः ।।
ऋत्वङ्गरस संज्ञास्युः षण्णां चापि प्रकीर्तिताः
मुनयोगिरिनामानि स्वरपर्याय एवच ।।
सप्तानां गणितं विद्याद्वसुश्च प्रकृतिस्तथा
नागानां चापिनामानि व्याख्यातानि विदुस्तथा ।।
अष्टानामथ रन्ध्रश्च सुषिरश्छिद्रगौरपि
नन्दशब्दो नवानां तु शास्त्रे/स्मिन् कथितानि तु ।।
दशानां पङ्क्तिसंज्ञा स्यात् दिगित्येतत्तु
कीर्त्यते रुद्राणां भास्कराणां च विश्वेदेवगणस्यच ।।
मनूनां च सुरेशानां तिथिनामानि कीर्तिताः
एकादशादि पञ्चानामष्टिः षोडश कीर्तिताः ।।
अत्यष्टिरिति सप्तानां सदशानां प्रकीर्त्यते

धृतिरष्टादशाख्या स्यात् एवमत्रोच्यते बुधैः ।।
आकाशस्य च नामानि असन्द्विन्दुश्च कीर्तिता
शून्यस्थानेषु सर्वेषु शास्त्रेऽस्मिन् पठिता बुधैः ।।
अनुक्तानां तु संख्यानां यद्दृष्टं तद्विचिन्त्य
वैकल्पनीयं बुधैरत्र प्रसिद्धं बहुषुश्रुतम् ।।

Chandra seethaamsurinduscha chandramaa himagu: sasee
evamaadeeni naamaani chandrasya kathithaani cha
roopamithyedathekasya dwayorapicha keerthyathe
nayanasya thu namaani yugmam yugalameva cha
yamam cha yamalam chaiva dasrow naasathya evacha
asvinornamadeyatvaath dvsankhyethi prakeerthithe
agninaamaani yaanyathra guno lokaascha pushkaraa:
raamo vratham thrayaanaamthu keerthithaani budaissadaa.
vedaparyaaya sabdascha samudrasya thadaiva cha
kruthaschethi chathurnaam cha sankhya sadbhirudaahruthaa
indriyaani cha bhoothaani kaamadeveshavasthathaa
vaayu paryaaya sabdaascha panchaanam thu prakeerthithaa:
munayo giri naamaani swaraparyaya eva cha
sapthaanaam ganitham vidyaadvasuscha prakruthisthathaa
naagaanaam chaapi naamaani vyakhyaathaani vidusthathaa
ashtaanaamatha randrascha sushiraschhidra gowrapi
nandasabdo navaanaam thu saasthresmin kathithaani thu
dasaanam pangthisamjna syaath digithyethatthu keerthyathe
rudraanaam bhaskaraanaam cha visvedevaganasyacha
manooonaam cha suresaanaam thithi naamaani keerthithaa:
ekaadasaadi panchaanaamashti: shodasakeerthithaa:
athyashtirithi sapthaanaam sadasaanaam prakeerthyathe
druthirashtaadasaakhyaasyaath evamathro chyathe budhai:
aakaasasya chanamaani asandvinduscha keerthithaa
sooonyasthaaneshu sarveshu sastresmin patithabudhai:
anukthaanaam thu sankhyaanaam yaddrushtam thadvichinthya
vaikalpaneeyam budhairathra prasiddham bahushusrutham

For Moon and its synonyms, roopam, etc value is one. Eyes, its synonyms, ears, yamam, yamalam, nose and aswini, denote number two. Fire, gunas, lokas and Rama denote three. Veda, occean, krutha, etc. denote four. Sense organs (organs in general) bhootha, Kamadeva and air denote five. Seasons, vedanga etc., stand for six. Muni, mountains, swara, etc denote seven. Vasu, serpent, prakruthi, knowledge are for eight. Openings (in human body), hole, nanda, etc. stand for nine. Dikh and pangthi give value of ten. Rudra, soorya, viswedeva, Manu and tithi, respectively stand for 11, 12, 13, 14 and 15. Ashti for 16, athyashti for 17, dhruti for 18, say the learned people. Synonyms of sky are used in the place of zero.

Synonyms of all the above nouns too carry the same value and they are commonly used in ancient books. Wherever required, based on rational thinking one can make numbers. Say for example the stars/ celestial bodies stand for 27, teeth (dantha) for 32, masa (month) for 12, Jina for 24, etc. For arriving at the above values, puranic/Vedic/historical/philosphical or observed facts can be taken as references.

Bhrahmagupta, Lallacharya, Bhaskaracharya (I &II) and many others have used this number system in all their scientific contributions. The advantage is that a number of synonyms can be used for writing the data in a poetical stanza style to keep the rhythm/ metre of presentation. Indu (moon) has the number value one. All of its synonyms like Chandra, Tharaanatha, Sasi, ... can conveniently be used for literary beauty of the poetic presentation, where the value 1 is required. An example on the application of bhootha sankhya system; *Vyoma - soonya - sara-adri-indu - ranthra - adri - adri-sara - indavaha* - denote the total number of days in a mahayuga (4320000 years) which is 1577917500, written in bhoothasankhya. A significant point to be noted is that when words are read from left to right, numbers

are written from right to left, as it is in modern mathematics. Also, each digit (word) carries the place value too.

Katapayadi number system : This is another system commonly used by astronomers and mathematicians in South India. In a detailed search of literature it is found that use of Katapayadi number was common in the books written by Kerala astronomers and mathematicians. As in bhoothasankhya, where synonyms are used for presenting mathematical data, the letters, in Katapayadi system, with or without vowels play the same role in presenting numbers. It was considered an art in Katapayadi system to make suitable Sanskrit words, using appropriate letters with or without vowels, to present numbers.

In this number system ka to jha, tha to dha (with or without vowels) carry 1 to 9 number values, respectively. Pa to bha (with or without vowels) 1 to 4 values. Ya to ha (with or without vowels) carry 1 to 8 value. Nja, na and all vowels used independently in the beginning are equal to zero. Vowels do not have any value if used with consonants or used independently in between, but at the beginning it stands for 0. In compound consonants the last consonant is to be taken for its number equivalent. Digits are written according to the letters, similar to bhootha sankhya system, i.e from right to left, with place values. Explanation on the use of Katapayadi system of notation is given in Sadratnamala, a book written by Sankaravarman, a famous Keralite mathematician astronomer. Example of the use of Katapayadi number system; *La ku tam* = 113; *A na ntha pu ram* = 21600. (A=0 na = 0 tha = 6 pu = 1 ra = 2)

Kerala Astronomers have used the combination of all the above number systems in their books. More applications of all these number systems are given in the mathematics and astronomy texts.

Discovery and use of Zero

While explaining the number and notations of mathematics, it is interesting to mention the development of numeral 0. It is accepted that zero was discovered by Indians. Pingalacharya's Chandasastra (200 BC) appears to be the first book in which application of Soonya is given for writing numbers as follows:

गायत्रे षड्संख्यामर्धे/पनीते द्वयङ्के अवशिष्ट स्त्रयस्तेषु
रूपमपनीय द्वयङ्काधः शुन्यं स्थाप्यम् ।।

Gaayathre shadsankhyaamardhef apaneethe dvayanke avasishtasthrayastheshu roopamapaneeya dvayankaadha: soonyam sthaapyam

In gayatri chandas, one pada has six letters. When this number is made half, it becomes three (i.e the pada can be divided into two). Remove one from three and make it half to get one. Remove one from it, thus gets the zero (Soonya).

Kane, P.V. in his article, on the decimal notation, which appeared in the journal of Bombay branch of the Royal Asiatic Society (I, 28, PP 159-60 (1953) says that "In view of the discovery of decimal place value concepts in India, it is accepted that 0 as a part of the numerical system is an Indian contribution. Pingalacharya's Chanda sastra first mentioned the word Soonya, describes the rules for calculating the number of long and short syllables in metre of different syllables".

Chanda sastra in the 29th stanza of the 8th chapter says : *Roope soonyam* and in 30th stanza it says: *Dvi soonyam*. Paulisa siddhanta (200 AD) and original Surya siddhanta (400 AD) have also used Soonya and kha (means sunyakasa = space = zero) Vyoma, akasha, ambara in the places of zero in bhootha sankhya.

Commentator Surya Deva has written : *'Kha'ni soonyopalakshithani* which means whenever kha is written it is for 0. खानि शून्योपलक्षितानि

As per the development of the numeral 0 (as it used now), it has undergone only minor changes during the process of its development. The Ancient Kashmiri book (first centruy BC) on Atharvaveda used big circular dotes, sufficiently large, for giving folio numbers. This was first observed by Maurice Bloomfield and Richard Garbe, of Baltimore in 1901, when they were reproducing the Vedic book into films. Bhakshali manuscripts also contain similar small circular dotes representing 0. From available literature one can conclude that the present form of 0 had its birth around the latter half of the first millennia BC, in India. During then, or a near later period using zero in other civilizations is thus explained by Needham, (Journal of Science & Civilization in China). "The Chinese, even upto the 8th century AD, left a gap or some vacant space similar to the Babylonians where 0 was required. A symbol for 0 in the usual circular form appeared only in 1247 AD in the Chinese work: Su Shu Chiu Chang of Chin-Chu-Shao". This refutes the stray claim put forward by some experts that 0 might have been developed in China, by assuming that the growth of astronomy and mathematics in China was at par with that of India during then. Albiruni (973 - 1048) writes that in Indian notation - when zero has to be written it does not have resemblance with ha or : (in Sanskrit).

Calculations with 0:

Results obtained when calculations are done with 0 were well known to ancient Indians. Sreedharacharya's Patiganitha (rule 21) says thus;

क्षेपसमं खं योगे राशिरविकृतः खयोजनापगमे
खस्य गुणनादिके खं संगुणने खेन च खमेव

Kshepasamam kham yoge raasiravikrutha: khayojanaapagame
khasya gunanaadhike kham sangunane khena cha khameva

When a number is added to cipher, the sum is equal to additives, when cipher is added to or subtracted from a number it remains unchanged. In multiplication and other operations, the result is cipher itself.

Rule for division is not given here separetely. Sreedharacharya only says that in other operations also the results are zero. However it is known that when a number is divided by 0, the result is infinity. This has been made very clear in Lilavati by Bhaskaracharya II (page 71 rule 1)

योगे खं क्षेपसमं, वर्गादौ खं, खभाजितो राशि :
खहर: स्यात् खगुण: खं खगुणाश्चिन्त्यश्च शेषविधौ।

Yoge kham kshepasamam, vargaadow kham, khabhaajitho raasi:
khahara: syaath khaguna: kham khagunaaschinthyascha seshavidhow

When 0 is added to a number, value is the number itself. Square of 0 is 0. When a number is divided by 0, the result is infinity (khahara). When multiplied by 0, it becomes 0. Think for their calculation like this.

Bhaskaracharya II, had defined the word Khahara as an endless number equal to infinity (*Anantho rasi*: infinity). Kha (0) hara (dividing) means 'that is divided by zero'.

Sripathi (1039 AD) in Siddhanta sekhara (14-6) gives explanation for the operation with 0 in a better way with more details and clear rules.

विकारमायान्ति धनऋणखानि न शून्य संयोग वियोगतस्तु
शून्याद्धि शुद्धं स्वमृणं क्षयं स्वं वधादिना खं खहरं विभक्ता:

Vikaaramaayaanthi dhanarunakhaani
na soonya samyoga viyogathasthu
soonyaaddhi suddham swamrunam kshayam
swam vadhaadinaa kham khaharam vibhakthaa:

Nothing happens (to the number) when a positive or negative number is added with 0. When +ve and -ve numbers are subtracted from 0, the +ve number becomes negative and -ve number becomes +ve. When multiplied with 0, the values of both +ve and -ve numbers become 0, when divided by 0, it becomes infinity (khahara).

Knowledge about infinity : As mentioned above, infinity was well known for ancient Indian mathematicians. Kha hara is the term used to represent infinity. It is also well defined literarally as *ananthо raisi* i.e never ending number. Bhaskaracharya II and others have given clear explanation with examples for infinity. In Bhaskaracharya's Beejaganitha (stanza 20), infinity is thus glorified.

अस्मिन् विकार: खहरे न राशावपि प्रवेष्टेष्वपि नि:सृतेषु
बहुष्वपि स्याल्लयसृष्टिकालेनन्तेच्युते भूतगणेषु यद्धत्
Asmin vikara khahare na raasaavapi praveshteshvapi ni:
srutheshu bahushvapi syaallaya srushtikaalef nanthef chyuthe
bhoothaganeshu yaddhath

Nothing happens to the (huge number) infinity, when any number enters (added) or leaves (subtrated) the infinity. During pralaya many things get dissolved in Mahavishnu and after pralaya, during srushti all those things get out of him. This happens without affecting the lord himself. Like that, whatever number is added to infinity or whatever is subtracted from it, the infinity remains unchanged.

This explanation for infinity is in full agreement with modern mathematics.

Positive and negative numbers and their calculations:

In ancient India knowledge of these fundamental rules on positive and negative numbers were at par with that of present day. Mention has been made about it, while discussing the zero. Brahmagupta and Bhaskaracharya II have given these aspects as early as the 6th century AD. Brahmagupta has given three rules on +ve and -ve numbers in Brahmasphuta siddhanta.

ऋणमृणयोर्धनयोर्घातो धनमृणयोर्धनवधो धनं भवति

Runamrunayordhanayorghatho
dhanamrunayordhanavadho dhanam bhavati

When two -ve numbers are multiplied, the resulting number is positive. When a +ve and a -ve number are multiplied, the result is -ve. When two +ve numbers are multiplied the result is +ve.

Later in 1150 AD Bhaskaracharya II has given the rules of +ve and -ve numbers in Beejaganitham, which is a chapter incorporated in his famous astronomical treatise Siddhanta siromani. Sree Krishna Daivajna has written a detailed commentary for Beejaganitham in 1650 AD. Rules for addition and subtration of +ve and -ve numbers are given in Beejaganitham (1.1)

योगे युतिः स्यात् क्षययो स्वयोर्वा धनर्णयोरन्तरमेव योगः

Yoge yuthi: syath kshayayo swayorvaa
dhanarunayorantharameva yoga:

One can add and subtract +ve and -ve numbers. The difference between the numbers gives sum of the +ve and -ve numbers.

Krishna Daiwajna, in his commentary on Beejaganitham, gives examples for above calculations with +ve and -ve numbers (1.1).

अत्रप्रथमोदाहरण: -3+-4 योगे जातम् -7 द्वितीये व्यास: 3+4 योगे जातम् +7. तृतीयेन्यास: +3+-4 जातम् -1 चतुर्थ न्यास: 4+-3 योगे जातम् +1

When -3 and -4 are added, the result is -7; secondly when +3 and +4 are added, it is 7, thirdly +3 and -4 are added, it is -1; fourthly +4 and -3 are added, result is 1.

Like this there are three/four types of calculations using +ve and -ve numbers. Says Krishna Daiwajna, in his commentary to the above (1.1)

रूपत्रयं रूपचतुष्टयं च धनं वा सहितं वदाशु

Roopathrayam roopachathushtayam cha
dhanam vaa sahitham vadaasu

Rule for the multiplication of +ve and -ve numbers in Beejaganitham (1.2)

स्वयोरस्वयो: स्वं वध: स्वर्णघाते क्षय:

Swayorasvayo: swam vadha: swarunaghate kshaya:

Among +ve and -ve numbers, when multiplied each other, the result is -ve. Rule of multiplying +ve numbers and -ve numbers among themselves in Beejaganitham (2.2)

धनं धनेनर्णमृणेन निघ्नं द्वयं धनं

Dhanam dhanenarnamrunena nighnam dvayam dhanam

When +ve number multiplied with a +ve number and -ve with a -ve number, the results are +ve numbers.

Krishna Daiwajna makes it clear that, when -ve numbers are multiplied the result is +ve.

अस्वयोर्वध स्वम् *Asvayorvadha swam*

When non +ve numbers are multiplied the +ve number is the result.

Rule for division of +ve and -ve numbers is also given in Beejaganitha (3.2)

भाज्य भाजकयोरुभयोरपि धनत्वे ऋणत्वे कालब्धिर्धनमेव

Bhaajya bhaajakayorubhayorapi
dhanathve runathve kalabdhirdhanameva

+ve and -ve numbers when divided among themselves the results are +ve.

Bhaskaracharya had, not only given the calculations with -ve and +ve numbers among themselves, but also had given rules for determining the squares and square roots of +ve and -ve numbers (Beejaganitham 1.4)

कृति स्वर्णयो: स्वं स्वमूले धनर्णे न मूलक्षयस्यास्ति तस्याकृतित्वात्

Kruthisvarunayo: swam swamoole dhanarne
na moolakshayasyaasthi thasyaakruthithwaath

For +ve and -ve numbers, square is always +ve. Because of the nature of -ve numbers there is no square root (for -ve numbers).

It is interesting to note that the ancient Indians could say that there is no real number as square root for a negative number. In modern mathematics it is an imaginary number. The same explanation is given using the words 'absence of shape': *akrutitwat.*

Number place :

The commonly used number writing method, using number places, is an ancient Indian contribution. I.e. the decimal system. As mentioned in the beginning of the text while discussing

number systems the decimal places were well known to ancient Indians from the vedic period onwards. They have defined the place values such as 1st, 10th 100th, 1000th,.......... etc. Ancient Egyptians, Babylonians, Arabs and Chinese followed entirely different systems for writing numbers. Egyptians and Semitic groups followed a method for writing numbers which is in Hieroglyphics system. Separate symbols were used for one, ten, hundred, ...etc. For example, the number 23 is written as 111UU = 1+1+1+10+10.

In Shang Oroacle bone form of Chinese system, separate symbols for one, five to ten, twenty, hundred, etc. were used. For the numbers two, three and four, one is repeated as we write iii for three. Similarly digits were repeated when symbols are not present. Without much improvement the same system was used by the Chinese even upto the 5th century AD. In Greek too, various symbols depicting numbers were used. For writing a larger number, addition procedure was followed in all the non Indian systems. Romans also used symbols for writing numbers and addition method was followed instead of place values.

The number 1548, in Roman system, is written MCCCCCXXXXVIII i.e 1000 + 500 + 40 + 5 + 1 + 1+ 1.

In India the use of symbols without the use of decimal value could be seen in Kharoshthi (250 BC) and Brahmi (250 BC) inscriptions. It disappeared by the 3rd century AD. Inscriptions in Nanaghat (150 BC) and that in Nasik (150 AD) caves and a few other inscriptions namely Kusana (150 AD), Kshatrapas (200 AD) resemble each other where decimal places are clearly used. These numbers are the basis of our modern numeral systems. There is no evidence of the use of the above system of numerals including zero in a decimal place value scale, anywhere in the world, as it was found in India. At the same time the system of

numbers with nine symbols from one to nine and the symbol zero, following the basis of decimal places value concept was used in ancient India. According to Sarton "our numerals and the use of zero were invented by the Hindus and transmitted to us through the Arabs and hence known as Arabic numerals which we have given them the name".

Explanations on place values are given by many authors. Aryabhatta says, after mentioning one, ten, hundred, ... upto more than a million, " .... from place to the next place it is the multiple of ten". The place values for digits in a number are mentioned very clearly in Vyasa Bhashya for Yogasutra which is historically an old book written not later than 650 AD. In this book (3-13) place values are thus illustrated.

A woman may be called mother (by her children), daughter (by her father) daughter-in-law (by her mother in law), even though she is the same woman. Similarly even if the digit is the same, depending upon its place, in the number, its value varies. In the unit place the digit has the same value, in 10th place, 10 times the value and in 100th place 100 times the value, is given.

यथा एकरेखा शतस्थाने शतं दशस्थाने दशैवं चैकस्थाने यथा च
एकत्वेपि स्त्री-माता च उच्यते दुहिता स्वसा च इति

Yathaa ekarekhaa sathasthaane satham dasasthane dasaiam chaikasthaane yathaa cha ekathvepi sthree mathaa cha uchyathe duhithaa svasaa cha ithi

Adi Sankaracharya (700 AD) in Vedanta sutra bhashya (II.2.17) has given similar explanation for writing numbers giving decimal place values.

यथाचैकापि रेखा स्थानान्यत्वेन निविशमानैक
दश शत सहस्रादि शब्द प्रत्यय भेदमनुभवति

Yathaachaikaapi rekha sthaananyathvena nivisamaanaika dasa satha sahasraadi sabda prathyaya bhedhamanubhavathi

One and the same numerical sign when occupying different places is conceived as measuring 1, 10, 100, 1000 etc.

Hence writing the numbers using the decimal places, is an Indian contribution. From the period of Rig Veda, Sanskrit number system was also common here. The Sanskrit number system was followed in the earlier mathematical treatise of Sulbasutras. Hence number places have also been followed.

In Manava Sulbasutra (9.4) the Sanskrit number is given:

नवांगुलसहस्राणि द्वे शते षोडशोत्तरे

Navangula sahasraani dve sathe shodasotthare

The fire altar measures 9216 angula, 9 thousand, 2 hundred and 16.

एकैकस्य सहस्रं स्याच्छते षण्णवतिः परा

In stanza (9.5) such a value is given: *Ekaikasya sahasram syacchatam shannavatihi para*: The area is 1196, 1 thousand, 1 hundred, ninety and six.

एकादश सहस्राणि अङ्गुलानां शतानि षट्
शतम् चैव सहस्राणां क्षेत्रमग्नेर्विधीयते

Ekaadasa sahasraani angulaanaam sathaani shad
satham chaiva sahasraanaam kshethramagnervidheeyathe

Area of fire altar is given in stanza (9.6) Area is 111600 sq. Angulas. The period and other details of Sulba sutras are discussed separately. Even in this oldest book one can observe the application of the decimal systems.

In the bhootha sankhya and katapayadi systems also, the number places are well defined and applied. Since bhootha sankhya was known at least as early as 200 BC, it can be assumed that application of number places while writing the numbers, was prevalent during the period.

Use of fractions in mathematical calculations:

The measurements of length, breadth and areas of bricks, sacrificial altars and halls (Yajnasalas) are given in whole numbers, fractions and even in square roots. A few examples are quoted in the geometrical part of this text.

Almost all ancient Indian mathematical texts directly use the fractions in the common application. Bhaskaracharya I in his commentary gives this quotation (123.1)

अर्ध षष्ठं द्वादशभागं चतुर्थभागसंयुक्तं ।
एकत्र कियद्द्रव्यं निर्देश्यं तत्क्रमेणैव ।।

Ardhashashtam dwaadasabhaagam chathurthabhaaga samyukthan ekathra kiyaddravyam nirdesyam thatkramenaiva

$\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{12}$ and $\frac{1}{4}$ are respectively added together, say what is the aggregate?

Tantrasangraha gives this rule for the addition and subtration of the fractions.

कुर्यात् समच्छिदामेव राशीनां योगमन्तरम्।।

Kuryaat samachidaameva raseenaam yogamantharam

For addition or subtraction, the denominators are to be equalised in a fraction.

In Aryabhateeya bhashy, Bhaskaracharya I (123.3) has given this exercise question, with fractions.

अर्धषड्भागोनं पञ्चांशच्चापि सप्तभागानाम् ।
त्र्यंश पादोनं वा गणयत कियद् द्रव्यम् ।।

Ardhashadbhagonam panchamsachhapi sapthabhaagaanaam vyamsa paadonam vaa ganayatha kiyad dravyam

Calculate O! mathematician what the sum amount to (when added together) $\frac{1}{2}$, $\frac{1}{6}$ $\frac{1}{5}$ $\frac{1}{7}$ $\frac{1}{3}$ $\frac{1}{4}$?

Same principle, which was used earlier is adopted in modern mathematics also. By taking the LCM two or more denominators are equalised and further processed. Aryabhatta I has given the method for operation with fractions (Aryabhateeya 2-27)

छेदा परस्पर हता भवन्ति गुणकार भागहाराणाम् ।।

Chhedaa paraspara hathaa bhavanthi gunakaara bhaagahaaraanaam

The numerators and denominators of multipliers and divisors are multiplied by one another (to simplify the fractions and then to process for addition or subtraction).

Aryabhateeya (2-28) gives this method for the determination of LCM of the denominators of fractions.

छेदगुणं सच्छेदं परस्परं तत् स्वर्णत्वम् ।।

Chhedagunam sachhedam parasparam thath swarunathvam

Multiply the numerators as also the denominators of each fraction by the denominator of the other fraction; then the fractions are reduced to common denominator.

Making common denominator is like taking the lowest common multiples of two numbers (LCM). Same principle has been explained by Bhramagupta in Brahmasphuta siddhanta (XII.2) and Aryabhatta II in Maha-siddhanta (XV.13). Sreepati in Siddhanta sekhara (XIII.11) has gone deep into this subject. The applied example given by Bhaskaracharya II for the calculations with fractions in Lilavati (2-1) is an impressive one which can be cited for showing the capability of ancient mathematicians to deal with fractional numbers.

द्रम्मार्धत्रिलवद्वयस्य सुमते पादत्रयं यद्भवेत्
तत्पञ्चाँशक षोडशाँशचरण: संप्रार्थितेनार्थिना ।
दत्तो येन वराटका: कति कदर्येणार्पितास्तेन मे ब्रूहित्वं
यदि वेत्सि वत्स गणित जातिं प्रभगाभिधाम्

Drammaardha thrilavadvayasya sumathe paadathrayam yadbhaveth that panchaamsaka shoda saamsa charana: sampraarthithenaa-rthinaa datto yenavaraatakaa: kathi kadaryenarpithastena me broohithvam yadi vetsi vatsaganitha jaathim prabhagaabhidhaam

One man has given to a beggar fraction of 1 dramma (a unit of money). That fraction is one fourth of the one sixth of one fifth of the three fourth of the two third of the half of a dramma. Then tell how much kowdi (a unit fraction of the amount dramma) was given to the beggar?

Answer can be obtained by multiplying all the fractions with the number of Kowdi equal to dramma. i.e. 1/4 x 1/6 x 1/5 x 3/4 x 2/3 x ½ x number of kowdies for one dramma. The work of a mathematical genius is demonstrated in this problem! Here the product of fractions is the product of numerator divided by products of denominators. This methodology for determining the product of fraction is also explained in ancient Sanskrit books. Patiganita rule 33 (i) says:

प्रत्युत्पन्नफलं स्यादंशवधे छेदघात संभवते ।।

Prathyuthpannaphalam syaadamsavadhe chhedaghaatha sambhavathe

The product of a given fraction is obtained on dividing the product of numerators by the product of denominators.

Square, square root and their applications:

Needham, J., in Science and Civilization in China (pp 65-68) says that the Chinese tried to find out square root (in Chinese language square root is Khifang) by geometrical methods different from that of Aryabhatta and had hardly any success in formalising a rule upto the 12th century AD. Smith D.E in History of mathematics (2, p 148) says thus about the situation in Europe: "In Europe these methods (for finding out the square and square root) did not appear before Cataneo (1546 AD). He gave the method of Aryabhatta for determining the square root."

In India the knowledge on theoretical and applied part of square and square root was atleast as old as Sulba sutras. Uses of square root can be seen in the measurements of length, breadth and area of bricks and the sacrificial halls. Method for finding out square root of 2 and 3 are given in Boudhayana sulba sutra. Dwikarani and Trikarani are the common mathematical terms adopted for square root of 2 and 3. Karani is an earlier term used in all Sulba sutras which was replaced by vargamoolam in the later mathematics books, for square root.

Jaina book on mathematics, Uttaradhyayana sutra, written in 300 BC gives following terms on the mathematical calculations as varga, ghana, vargavarga (4th power) ghana varga (6th power) ghana varga varga (12th power) etc. Anuyogadwara sutra of 1st century BC gives, 1st square and 2nd square and also their square roots. Aryabhatta in Aryabhateeya (2.4, 5) has given a method for finding out square root of numbers having many digits:

भागं हरेदवर्गान्नित्यं द्विगुणेन वर्गमूलेन ।
वर्गाद्वर्गे शुद्धे लब्धम् स्थानान्तरे मूलम् ।।

Bhaagam haredavargaannithyam dvigunena vargamoolena
vargadvarge suddhe labdham sthaananthare moolam

(Having subtracted the greatest possible square from last odd place and then having written down the square root of the number subtracted in the line of the square root) always divide the even place (standing on the right) by twice the square root. Then, having subtracted the square (of the quotient) from the odd place (standing on the right) set down the quotient at the next place (i.e on the right of the number already written in the line of the square root). This is the square root. (Repeat the process if there are still digits on the right)

Kaye has stated that Aryabhatta's method is algebraic in character and that resembles the method given by Theon of

Alexandria. This statement was proved wrong by Clarke, W.E and his colleagues and they declared that Aryabhatta's method is perfect and novel. Similar method has been given by Aryabhatta for finding cube root, also.

अघनाद् भजेद्द्वितीयात् त्रिगुणेन घनस्य मूलवर्गेण ।
वर्गास्त्रिपूर्व गुणित: शोध्य: प्रथमाद् घनश्च घनात् ।।

Aghanaad bhajedwitheeyaath thrigunena ghanasya moola varghena
vargaastripoorva gunitha: soddhya: prathamaath ghanascha ghanaath

(Having subtracted the greatest cube from the last cube place and then having written down the cube root of the number subtrated in the line of the cube root) divide the second non cube place (standing on the right of the last cube place) by thrice the square of the cube root (already obtained) (then) subtract from the first non cube place (standing on the right of the second non cube place) the square of the quotient multiplied thrice the previous (cube root); and (then subtract) the cube (of the quotient) from the cube place (standing on the right of the first non cube place) and write down the quotient on the right of the previous cube root in the line of the cube root, and treat this as the new cube root. Repeat the process for larger numbers.

These are the two methods known for finding out the square root and cube root for numbers having any number of digits. The credit of discovering these two methods has to go to Aryabhatta I. Sreedharacharya gives an important definition for square in Patiganita (rule 24):

सदृशाद्विराशिघातो रूपादिद्विचय पदसमासो (वा) ।
इष्टोनयुतपदवधो वा तदिष्टवर्गान्वितो वर्ग: ।।

Sadrusaadviraasighaatho roopaadidvichaya padasamaaso (vaa)
Ishtonayutha padavadho vaa thadishtavargaanvitho varga:

Square is equal to the product of two equal numbers or the sum of as many terms of the series whose first term is 1 and common difference is 2 or the product of the difference and the sum of the given number and an assumed number plus the square of the assumed number.

These can be mathematically represented as:

$n^2 = n \times n$ or $1+3+5+....n$ terms or $(n-a)(n+a) + a^2$

Similar definition is also given in Ganitasara sangraha (II.29). Sreedharacharya in Patiganita (rule 118) gives a simple method for determining the approximate value of square root of small non squaring numbers:

राशेरमूलदस्याहतस्य वर्गेण केनचिन्महता ।
मूलं शेषेण विना विभजेद् गुणवर्गमूलेन ।।

Raaseramooladasyaahathasya vargena kenachinmahathaa
moolam seshena vinaa vibhaje gunavargamoolena

Of the non square number as multiplied by some large number (having square root) extract the square root, neglecting the remainder and divide that by the square root of the multiplier.

I.e. multiply the number with a large square number and find out the square root of the product. Neglecting remainder, divide that square root by square root of the multiplier. (This gives an approximate square root of small numbers). An example for finding the square root of 10, multiply it with 100 and find out approximate square root of 1000 as 31 (neglect the remainder) divide 31 by the square root of 100 to get 3.1. Approximate value of square root for 10 is obtained. Instead of 100, if large number is used better accuracy would be obtained for square root.

Patiganita (example 99) gives yet another interesting exercise to prove the depth down to which the ancient Indian mathematicians went in search of the application of square root

मूलं शेषात्षष्ट: शेषपदं शेषपञ्चमो दत्त: ।
राशे: शेषस्य पदद्वितयम् रूपाष्टकम् शिष्टम् ।।

Moolam seshaathshashta: seshapadam seshapanchamo dattha:
rase: seshasya padadvithayam roopaashtakam sishtam

A number is diminished by its square root, what remains is diminished by its one sixth, what remains after that is diminished by its square root, what remains after that is diminished by its one fifth and what remains after that is diminished by twice the square root of itself; the residue now left is 8. Find out the number.

Obviously this excellent mathematical problem will stand as a proof of the status of the use of square roots in the 7th century AD in India.

Square and square root of fractions:

Square of numerator divided by the square of denominator is the square of the fraction says Sreedharacharya in Patiganita (34.1)

अंशकृतौ भक्तायाँ छेदनवर्गेण भिन्नवर्गफलं ।

Amsakruthow bhakthaayaam chhedanavargena bhinnavarga phalam

Brahmasphuta siddhanta XII (ii) and Ganita sara sangraha (III.13) also give the same rule for finding square root of fractions. In Aryabhateeyabhashyam, Bhaskaracharya I gives this mathematical exercise for the calculation of square root of a fraction (52.2).

षण्णां सचतुर्थानां त्रयोदशानां (स) चतुर्नवांशानां ।
विगणस्य वर्गमूले वद भट्टसंख्यानुसारेण ।।

Shannaam sachathurthanaam thrayodaasaanaam (sa)
chathurnavaamsaanaam

viganasya varga moole vada bhatta sankhyanusaarena

Calculate in accordance with the Ganita of Bhatta, the square root of 6 + 1/4 and of 13 +4/9 and state the two results.

This mixed fraction is to be converted into simple fraction and square root of numerator and denominator are to be taken. Here Ganita of Bhatta means, following the method given by Aryabhatta I.

Cube and cube root:

Determination of cube and cube root of large numbers was in common practice in olden days. This becomes clear when we go through the methodology and mathematical exercise given in the Sanskrit text books. Aryabhatta thus explains about the mathematical meaning of cube (Aryabhateeya 2-3)

सदृशत्रयसंवर्गो घनस्तथा द्वादशाश्रिः स्यात् ।

Sadrusathraya samvargo ghanasthathaa dwadasasri: syaath

The continuous product of three equals as also the (rectangular) solid having 12 equal edges are called cube.

Similar definition is also given in the Brahmasphuta siddhanta (XVIII. 42), Ganitha sara sangraha (II. 43) and Siddhanta sekhara (XIII. 4) It is interesting that in modern mathematics too the term cube stands for two mathematical meaning as defined in the above stanza. Ghana, in Sanskrit means, a factor of power with the number, multiplied by itself thrice and also a cubical structure. Hence the use of square and cube for two different meanings is an Indian contribution.

Bhaskaracharya II in Lilavati (1-13) gives an interesting problem for the calculation of cubes and cube roots of a series of numbers in his usual style:

नवघनं त्रिघनस्य घनं तथा कथय पञ्च घनस्य घनं च मे ।
घनपदं च ततोऽपि घनात् सखे यदि घनेऽस्ति घना भवतो मतिः ।।

Navaghanam thrighanasya ghanam thadaa kathaya panchaganasya ghanam cha me ghanapadam cha thathofpi ghanaath sakhe yadi ghanefsthi ghanaa bhavatho mathi;

Tell me O! intelligent, the cube of 9, 3 and 27 and 5, 125 and also the cube root of all these.

Bhaskaracharya I (Bhaskarabhashyam to Aryabhaṭeeya 51.3) gives another excercise for calculating cubes of numbers having double powers.

एकादिनवान्तानां रूपाणां मे घनं पृथक् ब्रूहि ।
अष्टाष्टकवर्ग घनं शतपादकृते: कृतेश्चापि ।।

Ekaadi navaanthaanam roopaanaam ghanam pruthak broohi ashtashtakavarga ghanam sathapaadakruthe: krutheschaapi

Tell me separately the cubes of the integral numbers beginning with 1 and ending in 9 and also the cubes of $[(8x8)^2]^3$ and (25^2)

This problem is an excellent example to show the capability to write the numbers in powers in multiple levels. Such problem is given just to find out the cube root, and was common in those days (529 AD)

Cube and cube root of fractions:

Cube root of fractions have also been determined by the ancient mathematicians. Bhaskaracharya I in his commentary to Aryabhaṭeeya gives this problem (51.4):

षट्पञ्चदशाष्टानां तावद्भागैर्विहीनगणितानाम् ।
घनसंख्यां वद विशदं यदि घनगणिते मतिर्विशदा ।।

Shatpanchadasaashtaanaam thaavadbhaagir viheenaganithaanam

ghanasankhyaam vada yadi ghanaganithe mathirvisadaa

If you have clear understanding of cubing a number, say correctly the cubes of 6, 5, 10 and 8 as respectively diminished by 1/6, 1/5. 1/10 and 1/8.

Here the exercise is to determine the cube of 6-1/6 i.e 5-5/6, similarly 5-1/5, 10-1/10 and 8-1/8. The rule for calculating the answers for the above problem has been well defined by these scholars. In Mahasiddhanta [XV 17(1)] of Aryabhatta II and Ganitasara sangraha (III. 13) it is said that:

अंशस्यघनं विभजेच्छेदस्य घनेन ।

Amsasya ghanam vibhajechhedasya ghanena

Cube of the numerator divided by cube of the denominator gives cube of the fraction.

In Bhaskarabhashyam, a problem for the determination of cube root is given (54.1)

एकादीनां मूलं घनराशीनां पृथक्तु मे ब्रूहि ।
वस्वश्विमुनीन्दूनां घनमूलं गण्यतामाशु ।।

Ekaadeenam moolam ghanaraaseenaam pruthakthu me broohi
vasvaswimuneendoonaam ghanamoolam ganyathaamaasu

Tell me separately the cube root of the cube number 1, etc (2, 3, 4, ...etc.) and also calculate the cube root of 1728.

Bhaskarabhashyam (54.2) for Aryabhateeya gives another exercise for determining the cube root of a larger number.

कृतयमवसुरन्ध्ररसाब्धिरूपरन्ध्रश्विनाग सङ्ख्यस्य ।
मूलं घनस्य सम्यक् वद भट्टशास्त्रानुसारेण ।।

Kruthayamavasurandravasaabdiroopa randrasvinaga sankhyasya
moolam ghanasya samyakvada bhatta sasthraanusarena

Correctly state in accordance with the rules prescribed in the Bhatta sastra the cube root of 8291469824.

This knowledge for calculating the cube root of large number is a noteworthy quality of ancient Indian mathematicians. Another complex calculation:

मूलं त्रयोदशानां पञ्चघनांशैस्त्रिशून्यरूपाख्यैः ।
अधिकानां भिन्नाख्यं विगण्यतां सङ्ख्यया सम्यक् ।।

Moolam thrayodasaanaam panchaghanamsai sthri sooonyarooopaakhyai:
Adhikaanaam bhinnaakhyam viganyathaam sankhyayaa samyak

Correctly calculate in accordance with the Ganita (of Aryabhatta) the fractional cube root of 13 +/-103/125.

This problem is also given in Bhaskarabhashyam (54.3) Here, two complicated mixed fractions are given (indirectly) for finding out the cube root. The fraction 103/125 is added to, in one of the figures and subtracted from other, and the cube roots are calculated for both mixed fractions. I.e. 13 + 103/125 and 13 - 103/125. An important point worth noting here is the use of +/ - sign in the same problem. It is generally assumed that simultaneous use of + and - in a mathematical statement is of later origin in the West.

Bhaskaracharya II in Lilavati gives this rule (1-13) on cube and square.

वर्गमूलघनः स्वघ्नो वर्गराशेर्घनो भवेत् ।

Varga moolaghana: swagnovargaraaserghano bhaveth

Take the square of the cube of the square root of a number, that is the cube of the number.

Let every student and teacher know about the true heritage of our motherland. Let us teach to others also.

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CONTENTS

Ancient Indian discoveries and their explanations on Numbers, Aryabhateeya number systems, bhootha sankhya systems, Katapayadi number systems, discovery of zero, calculations with zero, infinity, positive and negative numbers, number places, fractions, squares and square root, cubes and cube root.